Chapter 6

Linear Quadratic Optimal Control

6.1 Introduction

In previous lectures, we discussed the design of state feedback controllers using using eigenvalue (pole) placement algorithms. For single input systems, given a set of desired eigenvalues, the feedback gain to achieve this is unique (as long as the system is controllable). For multi-input systems, the feedback gain is not unique, so there is additional design freedom. How does one utilize this freedom? A more fundamental issue is that the choice of eigenvalues is not obvious. For example, there are trade offs between robustness, performance, and control effort.

Linear quadratic (LQ) optimal control can be used to resolve some of these issues, by not specifying exactly where the closed loop eigenvalues should be directly, but instead by specifying some kind of performance objective function to be optimized. Other optimal control objectives, besides the LQ type, can also be used to resolve issues of trade offs and extra design freedom.

We first consider the **finite time horizon case** for general time varying linear systems, and then proceed to discuss the **infinite time horizon** case for Linear Time Invariant systems.

6.2 Finite Time Horizon LQ Regulator

6.2.1 Problem Formulation

Consider the $m-$ input, $n-$state system with $x \in \Re^n$, $u \in \Re^m$:

$$\dot{x} = A(t)x + B(t)u(t); \qquad x(0) = x_0. \tag{6.1}$$

Find open loop control $u(\tau)$, $\tau \in [t_0, t_f]$ such that the following objective function is minimized:

$$J(u, x_0, t_0, t_f) = \int_{t_0}^{t_f} \left[x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\right] dt + x(t_f)^T S x(t_f). \tag{6.2}$$

where $Q(t)$ and S are symmetric positive semi-definite $n \times n$ matrices, $R(t)$ is a symmetric positive definite $m \times m$ matrix. Notice that x_0, t_0, and t_f are fixed and given data.

The control goal generally is to keep $x(t)$ close to 0, especially, at the final time t_f, using little control effort u. To wit, notice in (6.2)

- $x^T(t)Q(t)x(t)$ penalizes the transient state deviation,
- $x^T(t_f)Sx(t_f)$ penalizes the finite state

- $u^T(t)R(t)u(t)$ penalizes control effort.

This formulation can accommodate regulating an output $y(t) = C(t)x(t) \in \Re^r$ at near 0. In this case, one choice for S and $Q(t)$ are $C^T(t)W(t)C(t)$ where $W(t) \in \Re^{r\times r}$ is symmetic positive definite matrix.

6.2.2 Solution to optimal control problem

General finite, fixed horizon optimal control problem: *For the system with fixed initial condition,*

$$\dot{x} = f(x,u,t); \qquad x(t_0) = x_0 \ \textit{given},$$

and a given time horizon $[t_0, t_f]$, *find* $u(t)$, $t \in [t_0, t_f]$ *such that the following cost function is minimized:*

$$J(u(\cdot), x_0) = \phi(x(t_f)) + \int_{t_0}^{t_f} L(x(t), u(t), t)dt$$

where the first term is the final cost *and the second term is the* running cost*.*

Solution:

$$\dot{\lambda} = -H_x = -\frac{\partial L}{\partial x} - \lambda^T \frac{\partial f}{\partial x} \tag{6.3}$$

$$\dot{x} = f(x,u,t) \tag{6.4}$$

$$H_u = -\frac{\partial L}{\partial u} - \lambda^T \frac{\partial f}{\partial u} = 0 \tag{6.5}$$

$$\lambda^T(t_f) = \frac{\partial \phi}{\partial x}(x(t_f)) \tag{6.6}$$

$$x(t_0) = x_0. \tag{6.7}$$

This is a set of $2n$ differential equations (in x and λ) with split boundary conditions at t_0 and t_f: $x(t_0) = x_0$ and $\lambda^T(t_f) = \phi_x(x(t_f))$, and an equation that would typically specify $u(t)$ in terms of $x(t)$ and/or $\lambda(t)$. We shall see the specialization to the LQ case soon.

Proof: The solution is obtained by converting the constrained optimal control problem into an unconstrained optimal control problem using the Lagrange multiplier *function* $\lambda(t) \in \Re^n$:

$$\bar{J}(u, x_0) = J(u(\cdot), x_0) + \int_{t_0}^{t_f} \lambda^T(t)[f(x,u,t) - \dot{x}]dt.$$

Note that $\frac{d}{dt}(\lambda^T(t)\dot{x}(t)) = \dot{\lambda}^T(t)x(t) + \lambda^T(t)\dot{x}$. So

$$\int_{t_0}^{t_f} \lambda^T \dot{x}\ dt = \lambda^T(t_f)\dot{x}(t_f) - \lambda^T(t_0)\dot{x}(t_0) - \int_{t_0}^{t_f} \dot{\lambda}^T x\ dt.$$

Let us define the so called Hamiltonian function $H(x,u,t) := L(x,u,t) + \lambda^T(t)f(x,u,t)$. Thus,

$$\bar{J} = \phi(x(t_f)) - \lambda^T(t_f)x(t_f) + \lambda^T(t_0)x(t_0) + \int_{t_0}^{t_f} \left[H(x(t), u(t), t) + \dot{\lambda}(t)x(t)\right]\ dt$$

The necessary condition for optimality is that the variation $\delta\bar{J}$ of the modified cost with respect to all feasible variations $\delta x(t)$, $\delta\lambda(t)$, $\delta u(t)$ and $\delta\lambda(t_f)$ should vanish.

$$\delta \bar{J} = [\phi_x - \lambda^T]\delta x(t_f) + \lambda^T(t_0)\delta x(t_0) + \int_{t_0}^{t_f} \left\{[H_x + \dot{\lambda}^T]\delta x(t) + [H_u]\delta u(t)\right\} dt$$
$$+ \int_{t_0}^{t_f} \delta\lambda^T(t)[f(x(t), u(t), t) - \dot{x}]dt$$

Since $x(t_0) = x_0$ is fixed, $\delta x(t_0) = 0$. Otherwise, other variations $\delta x(t)$, $\delta u(t)$ or $\delta\lambda(t)$ are all feasible. Setting the terms that multiply these variations to be zero yield Eqs.(6.3)-(6.6). $\diamond$

6.2.3 Open loop solution

Applying the general optimal control solution in section 6.2.2 to the LQ problem in Eqs.(6.1)-(6.2), we have:

Theorem 6.2.1 *The optimal control is given by:*

$$u^o(t) = -R^{-1}B^T(t)\lambda(t) \tag{6.8}$$

where $\lambda(t)$ and $x(t)$ satisfy the Hamilton-Jacobi equation:

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \underbrace{\begin{pmatrix} A(t) & -B(t)R^{-1}B^T(t) \\ -Q(T) & -A^T(t) \end{pmatrix}}_{\textit{Hamiltonian Matrix - } H(t)} \begin{pmatrix} x \\ \lambda \end{pmatrix} \tag{6.9}$$

with boundary conditions:

$$x(t_0) = x_0; \qquad \lambda(t_f) = Sx(t_f). \tag{6.10}$$

- Boundary conditions are specified at both initial time t_0 and final time t_f (two point boundary value problem). In general, these are difficult to solve and require iterative methods such as *shooting method*.
- Optimal control in Eq. (6.8) is **open loop**. It is computed by first computing $\lambda(t)$ for all $t \in [t_0, t_f]$ and then applying $u^o(t) = -R^{-1}B^T(t)\lambda(t)$.
- Open loop control is not robust to disturbances or uncertainties.

6.2.4 Feedback control solution

Consider the *matrix* differential equation using the Hamiltonian matrix $H(t)$, where $X_1(t)$, $X_2(t) \in \Re^{n\times n}$.

$$\begin{pmatrix} \dot{X}_1(t) \\ \dot{X}_2(t) \end{pmatrix} = \underbrace{\begin{pmatrix} A(t) & -B(t)R^{-1}B^T(t) \\ -Q(T) & -A^T(t) \end{pmatrix}}_{\text{Hamiltonian Matrix - } H(t)} \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} \tag{6.11}$$

with boundary conditions $X_1(t_f) \in \Re^{n\times n}$ being any invertible matrix, and

$$X_2(t_f) = SX_1(t_f).$$

$X_1(t)$ and $X_2(t)$ can be integrated backwards in time from $t_f \to t_0$.

Let us assume (and it can be proven) that $X_1(t)$ is invertible. We propose that the solution to the Hamilton-Jacobi equation (6.9)-(6.10) is given by:

$$\begin{pmatrix} x(t) \\ \lambda(t) \end{pmatrix} = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} v$$

for some *constant* vector v.

$x(t)$ and $\lambda(t)$ as proposed clearly satisfy (6.9), and the boundary condition $\lambda(t_f) = Sx(t_f)$. The initial condition $x(t_0) = x_0$ can be satisfied by choosing $v = X_1^{-1}(t_0)x_0$.

If we define $P(t) = X_2(t)X_1^{-1}(t)$, then $\lambda(t) = P(t)x(t)$, so that the optimal control in Eq. (6.8) can be implemented as a feedback as given in the following theorem.

Theorem 6.2.2 *The cost function* (6.2) *is minimized using the control:*

$$u^*(t) = -R(t)^T B^T(t)P(t)x(t) \tag{6.12}$$

where $P(t) \in \Re^{n\times n}$ is the solution to the following so called continuous time Riccati Differential Equation (CTRDE):

$$-\dot{P}(t) = A^T(t)P(t) + P(t)A(t) - P(t)B(t)R^{-1}(t)B^T(t)P(t) + Q(t); \qquad P(t_f) = S. \tag{6.13}$$

Moreover, the minimum cost achieved using the above control is:

$$J^*(x_0, t_0, t_f) := min_{u(\cdot)} J(u, x_0) = x_0^T P(t_0)x_0$$

Proof: The feedback form of the optimal control Eq.(6.12) has already been shown. To show that CTRDE in Eq.(6.13) is satisfied by $P(t)$, one needs only differentiate $P(t) = X_1^{-1}(t)X_2(t)$, and making use of Eq.(6.11) and its boundary conditions.

The proof that $P(t)$ determines the minimal cost will be discussed later using Dynamic Programming (DP) principle. ◇

Remarks

1. $P(t)$ is solved backwards in time from $t_f \to t_0$ and should be stored in memory before use.

2. The optimal control law is in the form of a time varying linear state feedback $u(t) = -K(t)x(t)$ with feedback gain $K(t) := R(t)^T B^T(t)P(t)$. The open loop optimal control can be obtained, if so desired, by integrating (6.1) with the control (6.12). It is, however, much better to utilize feedback than to use openloop.

3. The Riccati differential equation can be derived from $P(t) = X_2(t)X_1^{-1}(t)$ and (6.11).

4. By direct substitution, it is easy to see the solution $\lambda(t) = P(t)x(t)$ satisfies (6.9)-(6.10). Since the solution of CTRDE (6.13) does not rely on solving for $X_1(t)$ or $X_2(t)$ explicitly, the assumption that $X_1(t)$ is invertible is in fact not needed for the proof of this theorem. It can be thought of as a useful device to guess the solution.

5. The control formulation works for time varying systems, e.g. nonlinear systems linearized about a trajectory.

6. $P(t)$ can be shown to be associated with the cost-to-go function (see below). Using this interpretation, it can easily be shown that $P(t)$ must be at least positive semi-definite.

6.2.5 Cost-to-go function

The matrix function $P(t)$ is associated with the so-called cost-to-go function. By this it is meant that if at any time $t_1 \in [t_0, t_f]$, and the state is $x(t_1)$, then, the control policy (6.12) for the remaining time period $[t_1, t_f]$ will result in a cost $J(u, x(t_1), t_1, t_f)$ in (6.2) with t_0 substituted by t_1 and x_0 substituted by $x(t_1)$ such that:

$$J^o(x(t), t, t_f) := min_u J(u, x(t), t, t_f) = x^T(t)P(t)x(t)$$

Since the optimal control, $u^o(t) = -K(t)x(t) = -R^{-1}(t)B^T(t)P(t)x(t)$, the closed loop system satisfies,

$$\dot{x} = [A(t) - B(t)K(t)]x(t)$$

so that $x(t) = \Phi(t, t_0)x_0$ where $\Phi(t, t_0)$ is the transition matrix for $A(t) - B(t)K(t)$. For this reason, the achieved minimal cost function must be of the form (omitting final time t_f to avoid clutter):

$$J^o(x_0, t_0) = J(u^o, x_0, t_0, t_f) = x_0^T \bar{P}(t_0)x_0.$$

for some positive semi-definite matrix $\bar{P}(t_0)$. Our task is to show that $\bar{P}(t_0) = P(t_0)$. To derive this result, we need the dynamic Programming (DP) Principle.

Dynamic Programming Principle

Consider the system:

$$\dot{x} = f(x(t), u(t), t), \qquad x(t_0) = x_0,$$

and the cost index over the interval $[t_0, t_f]$ is:

$$J(u(\cdot), x_0, t_0) = \int_{t_0}^{t_f} L(x(t), u(t), t)dt + \phi(x(t_f)). \tag{6.14}$$

In the theorem below, t_f is assumed to be fixed.

Theorem 6.2.3 *Suppose that $u^o(t)$, $t \in [t_0, t_f]$ minimizes (6.14) subject to $x^o(t_0) = x_0$ and $x^o(t)$ is the associated state trajectory. Let the (minimum) cost achieved using $u^o(t)$ be:*

$$J^o(x_0, t_0) = \arg \min_{u(\tau), \tau \in [t_0, t_f]} J(u(\cdot), x^o, t_0, t_f)$$

Then, for any t_1 s.t. $t_0 \le t_1 \le t_f$, the restriction of the control $u^o(\tau)$ to $\tau \in [t_1, t_f]$ minimizes

$$J(u(\cdot), x^o(t_1), t_1) = \int_{t_1}^{t_f} L(x(t), u(t), t)dt + \phi(x(t_f))$$

subject to the initial condition $x(t_1) = x^o(t_1)$. i.e. $u^o(\tau)$ is optimal over the sub-interval $[t_1, t_f]$.

Corollary 6.2.4 *Let $t_0 \le t_1 \le t_f$. Consider the optimal control problem for the sub-interval $[t_1, t_f]$. If $J^o(x_0, t_1)$ is the optimal cost and the optimal control is given by $u(t) = u^o(x_0, t)$ for $t \in [t_1, t_f]$. Then, the optimal control for the larger interval $t \in [t_0, t_f]$ with initial condition $x(t_0) = x_0$ is given by:*

$$u(t) = \begin{cases} \arg\min_{u(\cdot)} \int_{t_0}^{t_1} L(x, u, t)dt + J^o(x(t_1), t_1) & t \in [t_0, t_1) \\ u^o(x(t_1), t) & t \in [t_1, t_f] \end{cases} \tag{6.15}$$

where $x(t_1)$ is the state attained via the control $u(t)$ above.

Typical application of DP

A typical use of DP is in computing the optimal control policy utilizing Eq.(6.15).

- Consider a time grid $t_0 < t_1 < \ldots < t_n < t_f$.
- Solve the optimal control problem for the sub-interval $[t_n, t_f]$ with **arbitrary** initial states $x(t_n) = x$. Let the optimal control be denoted by $u(t) = u_n^o(x, t)$ and let the optimal cost given initial state $x(t_n) = x$ be denoted by $J^o(x, t_n)$. Here $u_n^o(x, t)$ for $t \in [t_n, t_f]$ and arbitrary initial state x is called the optimal control policy, and $J^o(x, t_n)$ is called the cost-to-go function at $t = t_n$.
- We now consider an iteration starting with $k = n$. Suppose that the optimal control for the initial time $t = t_k$ has been found and is given by: $u_k^o(x, t)$; and $J^o(x, t_k)$ is the cost-to-go function. Now consider initial time t_{k-1}.

  1. For each initial state, $x(t_{k-1}) = x$, compute, according to Eq.(6.15) in Corollary 6.2.4, the optimal control $u_{k-1}^o(x, t)$ for the interval $[t_{k-1}, t_f]$:

  $$u_{k-1}^o(x,t) = \begin{cases} \arg\min_{u(\cdot)} \int_{t_{k-1}}^{t_k} L(x,u,t)dt + J^o(x(t_k), t_k) & t \in [t_{k-1}, t_k) \\ u_k^o(x(t_k), t) & t \in [t_k, t_f] \end{cases} \tag{6.16}$$

     where $x(t_k)$ is the state achieved at $t = t_k$ from the initial state $x(t_{k-1})$ using optimal control $u^o(t, x(t_{k-1}), t_0)$ over the interval $[t_{k-1}, t_k]$.
  2. Compute the optimal cost $J^o(x, t_{k-1})$ for each x.
  3. Let $k \leftarrow k - 1$ and repeat from step 1 until $k = 0$.

- Notice that the optimal cost $J^o(x, t_k)$ is the cost-to-go function at time t_k.

Relating $P(t)$ to cost-to-go function for the LQ problem

Let us apply DP to the LQ case:

$$\begin{aligned} L(x,u,t) &= x^T Q(t) x + u^T R(t) u \\ \phi(x(t_f)) &= x^T(t_f) S x(t_f) \\ f(x,u,t) &= A(t)x + Bu \\ J &= \int_{t_0}^{t_f} L(x,u,t)dt + \phi(x(t_f)). \end{aligned}$$

Theorem 6.2.5 *The cost-to-go function for any $t \in [t_0, t_f]$ is given by:*

$$J^o(x,t) = x^T(t)\bar{P}(t)x(t)$$

where $\bar{P}(t) \equiv P(t)$ satisfies the Riccati difference equation Eq.(6.13) with boundary condition $\bar{P}(t_f) = S$. $P(t)$ is positive semi-definite for all $t \le t_f$. The optimal control policy is given by:

$$u^o(t) = -R^{-1}(t)B^T(t)\bar{P}(t_1)x(t)$$

Proof: At $t = t_f$, the cost-to-go function is simply:

$$J^o(x, t_f) = x^T S x = x^T \bar{P}(t_f) x$$

Hence, $\bar{P}(t_f) = S$.

Let $t_1 = t_f$ and consider $t = t_1 - \Delta t$ where $\Delta t > 0$ is infinitesimally small.

According to Eq.(6.15), the optimal control at t given the state $x(t)$ is obtained by minimizing

$$min_{u(t)} L(x, u, t)\Delta t + J^o(x(t_1), t_1)$$

Now, $x(t_1) \approx x(t) + [A(t)x(t) + B(t)u(t)]\Delta t$. Thus, we minimize w.r.t. $u(t)$,

$$\begin{aligned}
&\int_t^{t_1} \left[x(\tau)^T Q(\tau)x(\tau) + u^T(\tau)R(\tau)u(\tau)\right] d\tau + J^o\left(x(t_1), t_1\right) \\
&\approx \left[x(t)^T Q(t)x(t) + u^T(t)R(t)u(t)\right] \Delta t + J^o\left(x(t) + [A(t)x(t) + B(t)u(t)]\Delta t, t_1\right) \\
&\approx \left[x(t)^T Q(t)x(t) + u^T(t)R(t)u(t)\right] \Delta t + x(t)\bar{P}(t_1)x(t) + [x^T(t)A^T(t) + u^T(t)B^T(t)]\bar{P}(t_1)x(t)\Delta t \\
&\qquad + x^T(t)\bar{P}(t_1)[A(t)x(t) + B(t)u(t)]\Delta t
\end{aligned}$$

Setting the differential w.r.t. $u(t)$ to be 0, we get back the optimal control policy:

$$\begin{aligned}
&u^{oT} R(t) + x^T(t)\bar{P}(t_1)B(t) = 0 \\
&\Rightarrow u^o(t) = -R^{-1}(t)B^T(t)\bar{P}(t_1)x(t)
\end{aligned}$$

The updated optimal cost-to-go function is:

$$\begin{aligned}
J^o(x(t), t) \approx &\left[x(t)^T Q(t)x(t) + u^{oT}(t)R(t)u^o(t)\right] \Delta t + [x^T(t)A^T(t) + u^{oT}(t)B^T(t)]\bar{P}(t_1)x(t)\Delta t \\
&+ x^T(t)\bar{P}(t_1)[A(t)x(t) + B(t)u^o(t)]\Delta t + x(t)\bar{P}(t_1)x(t)
\end{aligned}$$

This shows that

$$\begin{aligned}
J^o(x(t), t) &\approx x^T(t)\bar{P}(t_1)x(t) + x^T(t)\left[A^T(t)\bar{P}(t_1) + \bar{P}(t_1)A(t)\right. \\
&\qquad \left. -\bar{P}(t_1)B(t)R^{-1}(t)B^T(t)\bar{P}(t_1) + Q(t)\right] x(t) \cdot \Delta t \\
&=: x^T(t)\bar{P}(t)x(t)
\end{aligned}$$

where

$$\bar{P}(t_1) = \bar{P}(t) + \left[A^T(t)\bar{P}(t_1) + \bar{P}(t_1)A(t) - \bar{P}(t_1)B(t)R^{-1}(t)B^T(t)\bar{P}(t_1) + Q(t)\right] \Delta t \tag{6.17}$$

Let $t \to t_1$, $\Delta t \to -dt$, and repeat the process and we get the update recursion in Eq.(6.17). Moreover, at each time t, $J^o(x(t), t) = x^T(t)\bar{P}(t)x$, a quadratic form as desired.

As $\Delta t \to 0$, Eq.(6.17) becomes:

$$-\dot{\bar{P}}(t) = A^T(t)\bar{P}(t) + \bar{P}(t)A(t) - \bar{P}(t)B(t)R^{-1}(t)B^T(t)\bar{P}(t) + Q(t);$$

which is exactly the Riccati differential equation as before. Hence $\bar{P}(t) = P(t)$.

Now, since

$$\begin{aligned}
x^T(t)P(t)x(t) &= \int_t^{t_f} \left[x^T(\tau)Q(\tau)x(\tau) + u^T(\tau)R(\tau)u(\tau)\right] d\tau + x^T(t_f)Sx(t_f) \\
&\geq 0
\end{aligned}$$

for any $x(t)$, $P(t)$ is positive semi-definite for any $t \leq t_f$. $\diamond$

6.3 Infinite time horizon case

Consider now the case when the system is time invariant, i.e. A, B in (6.1), and Q and R in (6.2) are constant matrices. Because of the infinite time horizon, the terminal cost condition is negligible. Thus, we assume that $S = 0$ and the cost function becomes:

$$J(u, x_0, t_0, t_f) = \int_{t_0}^{t_f} \left[x^T(t)Qx(t) + u^T(t)Ru(t)\right] dt. \tag{6.18}$$

Let us denote the solution of the CTRDE on the horizon $[t, t_f]$ by $P(t, t_f)$ where $t \in [t_0, t_f]$.

We are interested in finding the situation when $t_f \to \infty$. Since the system is time invariant, this is equivalent to fixing t_f and setting $t \to -\infty$. Three questions we need to ask are:

- For a fixed t_f, if we solve $P(t, t_f)$ in (6.13) backwards in time, does $P(t \to -\infty, t_f)$ exist (i.e. does it converge when $t \to -\infty$) ?
- If $\lim_{t\to-\infty} P(t, t_f) = \lim_{t_f\to\infty} P(t, t_f) = P_\infty$ does exist, there is a constant state feedback gain given by: $K = R^{-1}B^T P_\infty$, will the closed loop system:

  $$\dot{x} = (A - BK)x$$

  be stable?
- If $\lim_{t\to-\infty} P(t, t_f) = \lim_{t_f\to\infty} P(t, t_f) = P_\infty$ does exist, we know that it must satisfy $\dot{P}(t) = 0$, i.e.

  $$A^T P_\infty + P_\infty A - P_\infty BR^{-1}B^T P_\infty + Q = 0. \tag{6.19}$$

  which is called the Algebraic Riccati equation (ARE). In that case, which solution of the ARE does the asymptotic solution of (6.13) correspond to?

Proposition 6.3.1 *Let (A, B) be controllable (or just stabilizable). Then, there exists a positive definite matrix M such that for any $t < t_f$ and for all $x \in \Re^n$,*

$$x^T P(t, t_f)x < x^T Mx.$$

Moreover, $P(t \to -\infty, t_f) = P(t, t_f \to \infty)$ converge to a positive semi-definite matrix P_∞.

Proof: We sketch the proof for the (A, B) controllable.

Let Δt be an arbitrary fixed time interval. For any initial time $t < t_f - \Delta t$ and initial state x_0, we can design a control $u(\tau)$, $\tau \in [t, t + \Delta t]$ such that $x(t + \Delta t) = 0$; and $u(\tau) = 0$ for $\tau > t + \Delta t$. The cost associated with this control is finite and is independent of t. Since $P(t, t_f)$ is positive semi-definite, $x^T Px$ is bounded implies that P is bounded. By choosing different x_0, we can define a positive definite matrix M such that $x^T Mx$ is greater than the cost for the initial state x using the control thus constructed.

Secondly, for any $t_0 \le t_1 \le t_2$,

$$J(u, x_0, t_0, t_2) = \underbrace{\int_{t_0}^{t_1} \left[x^T Qx + u^T Ru\right] d\tau}_{J(u,x_0,t_0,t_1)} + x^T(t_1)P(t_1, t_f)x(t_1)$$

since $x^T(t_1)P(t_1, t_f)x(t_1)$ is the cost-to-go function. From this it can be seen that for any $t_0 \le t_1 \le t_2$,

$$J^o(x_0, t_0, t_2) = x_0^T P(t_0, t_2)x_0 \ge x_0^T P(t_0, t_1)x_0 = J^o(x_0, t_0, t_1)$$

i.e. the optimal cost increases as the time interval increases for the same initial condition. To see that this is true, suppose $x_0^T P(t_0,t_2)x_0 < x_0^T P(t_0,t_1)x_0$. Then, using the $[t_0,t_2]$ optimal control during the interval $[t_0,t_1]$ portion alone would achieve a cost over the interval $[t_0,t_1]$ that is less than $x_0^T P(t_0,t_1)x_0$. Since the latter is supposed to be the optimal, this is a contradiction.

This shows that for any $\Delta > 0$ and $x \in \Re^n$,

$$x^T M x \geq x^T P(t_0,t_2)x \geq x^T P(t_0,t_1)x$$

From analysis, we know that a non-decreasing, upper bounded function converges, thus, $x^T P(t,t_f+\Delta)x = x^T(t-\Delta,t_f)x$ converges as $\Delta \to \infty$. By choosing various x, a matrix P_∞ can be constructed s.t. for any x,

$$x^T P(t,t_f+\Delta)x \to x^T P_\infty x.$$

$\diamond$

Example To illustrate the necessity of (A,B) being stabilizable, consider a uncontrollable system

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}1\\ 0\end{pmatrix}u$$

with

$$J = \int_{t_0}^{t_f} [x_2^2 + u^2]dt$$

Since u has no influence on x_2, the optimal control is $u \equiv 0$ but $t_f \to \infty$

$$J^o(x,t_0,t_f) = \int_{t_0}^{t_f} x_2^2(t_0)e^{2(t-t_0)}dt \to \infty$$

$\blacksquare$

Next, we consider the stability question. The idea is that to ensure that the closed loop system is stable, one must ensure that all possible unstable behavior must be reflected in the performance index.

Proposition 6.3.2 *Let $Q = C^T C$ and suppose that (A,B) is stabilizable. If (A,C) is observable (or detectable), then the optimal closed loop control system*

$$\dot{x} = (A - BR^{-1}B^T P_\infty)x$$

is stable. Furthermore, P_∞ is strictly positive definite (positive semi-definite) if (A,C) is observable (detectable).

Proof: Suppose that (A,C) is detectable but the closed loop system is unstable. Let ν be the unstable eigenvector of $A - BR^{-1}B^T P_\infty$ such that

$$\lambda\nu = (A - BR^{-1}B^T P_\infty)\nu; \qquad Re(\lambda) > 0.$$

Let $x(t_0) = \nu$ be the initial state. Then, $x(t) = e^{\lambda(t-t_0)}\nu$. Since (A,B) is stabilizable, the cost is finite so that

$$\int_{t_0}^{\infty} x^T Q x\ dt < \infty; \qquad \int_{t_0}^{\infty} u^T R u\ dt < \infty$$

We assume that λ is real for simplicity. If λ is complex, we need to consider both λ and $\bar{\lambda}$ simultaneously. Then, since $e^{\lambda(t-t_0)} > 1$ for all $t - t_0 > 0$,

$$\int_{t_0}^{\infty} \nu^T Q \nu e^{2\lambda(t-t_0)} dt < \infty \quad \Rightarrow C\nu = 0.$$

$$\int_{t_0}^{\infty} u^T R u = \int_{t_0}^{\infty} \nu^T P B R^{-1} B^T P \nu e^{2\lambda(t-t_0)} dt < \infty \quad \Rightarrow R^{-1} B^T P_\infty \nu = 0.$$

This implies that

$$(A - BR^{-1}B^T P_\infty)\nu = A\nu = \lambda\nu$$

This contradicts the assumption that (A, C) is detectable, since,

$$\begin{pmatrix} \lambda I - A \\ C \end{pmatrix} \nu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence, (A, C) is detectable implies that the closed loop system is stable.

To show that P_∞ is strictly positive definite when (A, C) is observable, suppose that P_∞ is merely positive semi-definite so that,

$$x_0^T P_\infty x_0 = \int_{t_0}^{\infty} x^T C^T C x + u^T R u \;\; dt = 0$$

for some initial state $x(t_0) = x_0$. This implies that for all t, $u^T(t) R u(t) = 0$ or $u(t) = 0.$; and $Cx(t) = 0$. Or, for all t,

$$\dot{x} = Ax; \quad Cx = 0.$$

This is not possible if (A, C) is observable. $\diamond$

If (A, C) is merely detectable, P_∞ can be semi-definite only. Let ν be an unobservable eigenvector. Then, for $x(t_0) = \nu$, $u = 0$ is the optimal control and $x(t) = e^{\lambda(t-t_0)}\nu$ is the state trajectory. Thus,

$$\nu^T P_\infty \nu = \int_{t_o}^{\infty} x^T(t) C^T C x(t) \; dt = 0,$$

Example To illustrate the necessity for (A, C) detectable, consider the undetectable system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

with

$$J = \int_{t_0}^{t_f} [x_1^2 + u_1^2 + u_2^2] \; dt$$

For the initial condition of $(x_1, x_2) = (0, 1)$, the optimal control is $u_1(t) = u_2(t) = 0$ with an optimal cost of 0. However,

$$x_2(t) = x_2(0) e^{t-t_0} \to \infty \quad \text{as } t \to \infty.$$

Thus, the closed loop system is unstable. ■

The main result combining the above two propositions is given by the following:

Theorem 6.3.3 *For the time invariant system $\dot{x} = Ax + Bu$, initial condition $x(0) = x_0$, with (A, B) is stabilizable. Let the performance criteria be:*

$$J(u, x_0, t, t_f) = \int_t^{t_f} x^T(\tau)Qx(\tau) + u^T(\tau)Ru(\tau)d\tau \tag{6.20}$$

as $t_f \to \infty$, where $Q = C^TC \geq 0$ and $R > 0$ are positive semi-definite and definite respectively. In that case, the solution to (6.13) *$P(t, t_f)$ with $P(t_f, t_f) = 0$ satisfies:*

$$\lim_{t_f \to \infty} P(t, t_f) = \lim_{t \to -\infty} P(t, t_f) =: P_\infty,$$

exists and the optimal control is given by:

$$u(t) = -R^{-1}B^TP_\infty x(t). \tag{6.21}$$

Furthermore, if (A, C) is detectable, then the closed loop system $A - BR^{-1}B^TP_\infty$ is stable.

If (A, C) is observable, then P_∞ is positive definite. If (A, C) is only detectable, then P_∞ is merely positive semi-definite.

Remarks:

- S has been thrown out, because as $t_f \to \infty$ it is not important (at least for the (A, C) detectable case).
- If (A, B) is stabilizable, then the boundedness of $P(t, t_f)$ as $t_f \to \infty$ is automatic. This is because the cost J of using any control that steers the system to $x = 0$ in finite time is finite. This is so because the cost, given by $x_0^TP(t, t_f)x_0$, for using the optimal control should be even less. Since this is true for any arbitrary x_0, $P(t)$ must be finite also.
- The convergence of $\lim_{t_f \to \infty} P(t, t_f) \to P_\infty$ where P_∞ is some positive semi-definite matrix is guaranteed by the stabilizability condition. Specifically, $P(t, t_f)$ is finite for any t_f, and the fact that $x_0^TP(t, t_f)x_0 \leq x_0^TP(t, t_f + \Delta)x_0 = x_0^TP(t - \Delta, t_f)x_0$. The latter is due to the increasing nature of $J(u, x_0, t, t_f)$ in (6.20) as t_f increases.
- The convergence $\lim_{t_f \to \infty} P(t, t_f) \to P_\infty$ can be guaranteed by the more relaxed condition that (A, C) does not have any unobservable mode on the imaginary axis, and S is sufficiently large. (See Appendix of Goodwin LQ2-D.4). Furthermore, the closed loop system obtained using $u(t) = -R^{-1}B^TP_\infty$ would also stable.
- Given modest assumptions (stabilizability and detectability), LQ methodology automatically generates a state feedback controller that is stable.

The need for the detectability assumption is to ensure that the optimal control computed using the $lim_{t_f \to \infty} P(t, t_f)$ generates a feedback gain $K = R^{-1}B^TP_\infty^s$ that stabilizes the plant, i.e. the eigenvalues $A - BK$ lie on the open left half plane.

One can easily see that if there is an unstable mode that is not observable (i.e. not detectable) then the optimal control will choose not to do anything about it (since it is not reflected in the performance criteria). Therefore, if (A, C) does not have unobservable mode on the imaginary axis, then for $P(t) \to P_\infty$, we need to re-insert a sufficiently large final penalty $x^T(t_f)Sx(t_f)$ in (6.20) with $S > P_\infty$.

6.4 Discrete time LQ problem

In passing, we mention that there is an equivalent theory for discrete time systems. For the system,

$$x(k+1) = A(k)x(k) + B(k)u(k); x(0) = x_0.$$

with an equivalent performance criteria:

$$J = x^T(k_f)Sx(k_f) + \sum_{k=k_0}^{k_f-1} \left[x^T(k)Q(k)x(k) + u^T(k)R(k)u(k)\right]$$

then the optimal control is given by:

$$u(k) = -\underbrace{[R(k) + B^T(k)P(k+1)B(k)]^{-1}B^T(k)P(k+1)A(k)}_{K(k)}\,x(k) \tag{6.22}$$

where $P(k)$ is given by the the discrete time Riccati difference equation (DTRDE):

$$P(k) = A^T(k)P(k+1)A(k) + Q(k) - A^T(k)P(k+1)B(k)[R(k) + B^T(k)P(k+1)B(k)]^{-1}B^T(k)P(k+1)A(k)$$

with boundary condition $P(k_f) = S$.

The optimal cost-to-go is: $J^o(x,k) = x^T P(k)x$.

For the infinite horizon ($k_f \to \infty$) LQ regular problem, consider the time invariant case where

$$x(k+1) = Ax(k) + Bu(k)$$

and $Q(k) = Q$ and $R(k) = R$ are constant positive semi-definite, and positive definite matrices. The P_∞ matrix satisfies the discrete time Algebraic Riccati Equation (ARE):

$$A^T PA - P - A^T PB[R + B^T PB]^{-1}B^T PA + Q = 0.$$

$$u(k) = -\underbrace{[R + B^T P_\infty B]^{-1}B^T P_\infty A}_{K}\,x(k) \tag{6.23}$$

If (A,B) is stabilizable, then the closed loop system is stable, meaning that the eigenvalues of $A - BK$ with K given in (6.22) have magnitudes less than 1 (lie in the unit disk centered at the origin).

6.5 Eigenvalue placements

LQR can be thought of as a way of generating stabilizing feedback gains. However, exactly where the closed loop poles are in the LHP is not clear. We now propose a couple of ways in which we can exert some control over them. The idea is to transform the problem.

In this section, we assume that (A,B) is controllable, and (A,C) is observable where $Q = C^T C$.

6.5.1 Guaranteed convergence rate

To move the poles so that they are at least to the left of $-\alpha$ (i.e. if the eigenvalues of $A - BK$ are λ_i, we want $Re(\lambda_i) < -\alpha$, hence more stable), we solve an alternate problem. Since

$$\dot{x} = A'x + Bu \rightarrow Re\left(\text{eig}(A' - BK)\right) < 0$$

Thus, setting $A' = A + \alpha I$, we solve the LQ problem for the plant:

$$\dot{x} = (A + \alpha I)x + Bu.$$

This ensures that the eigenvalues of $Re((A+\alpha I)-BK) < 0$. Notice that $(A+\alpha I)-BK$ and $A-BK$ have the same eigenvectors. Thus, the eigenvalues of $A - BK$, say λ_i and those of $A + \alpha I - BK$, σ_i, are related by $\lambda_i = \sigma_i - \alpha$. Since $Re(\sigma_i) < 0$, $Re(\lambda_i) < -\alpha$.

6.5.2 Eigenvalues to lie in a disk

A more interesting case is to ensure that the eigenvalues of the closed loop system lie in a disk centered at $(-\alpha, 0)$ and with radius $\rho < \alpha$. This, in addition to specifying the convergence rate to be faster than $\alpha - \rho$, it also specifies limits for the damping ratio, so that the system will not be too oscillatory.

The idea is to use the discrete time LQ solution, which ensures that the eigenvalues of $A - BK$ lie in a unit disk centered at the origin. We need to scale the disk and to translate it. Let the continuous time plant be:

$$\dot{x} = Ax + Bu$$

- If we solve the discrete time LQ problem for the plant,

  $$x(k+1) = \frac{1}{\rho}A'x(k) + \frac{1}{\rho}Bu(k)$$

  then, the eigenvalues of $\frac{1}{\rho}(A'-BK)$ would lie in the unit disk and the eigenvalues of $(A'-BK)$ would lie in the disk with radius ρ, both centered at the origin.

- Using the same trick as before, we now translate the eigenvalues by $-\alpha$ by setting $A' = A+\alpha I$.

In summary, if we use the discrete time LQ control design method for the plant

$$x(k+1) = \frac{1}{\rho}(A + \alpha I)x(k) + \frac{1}{\rho}Bu(k)$$

then, the eigenvalues of $\frac{1}{\rho}((A + \alpha I) - BK)$ would lie within the unit disk centered at the origin. This implies that the eigenvalues of $((A+\alpha I) - BK)$ lie in a disk of radius ρ centered at the origin. Finally, this implies that the eigenvalues of $A - BK$ lie in a disk or radius ρ centered at $(-\alpha, 0)$.

6.6 Selection of Q and R

The quality of the control design using LQ method depends on the choice of Q and R (and for finite time S also). How should one choose these? Normally, this requires some kind of trial and error.

- Generally an iterative design/simulation process is needed;

- If there is a specific output $z = Cx$ that need to be kept small, choose $Q = C^T C$.
- Use physically meaningful state and control variables and use physical insights to select Q and R.
- Choose Q and R to be diagonal in the absence of information about coupling.
- Obtain acceptable excursions:

$$|x_i(t)| \leq x_{i,max}, \quad |u_i(t)| \leq u_{i,max}, \quad |x_i(t_f)| \leq x_{i,f-max}$$

  Then choose Q, R and S to be inversely proportional to $x^2_{i,max}$, $u^2_{i,max}$ and $x^2_{i,f-max}$ respectively.
- Off diagonal terms in Q reflect coupling. e.g. to coordinate $x_1 = -kx_2$, one can choose $C = [1 \;\; k]$ so that $Q = \begin{pmatrix} 1 & k \\ k & k^2 \end{pmatrix}$. One can add other objectives to Q.
- For finite time regulator problem with time interval T. The ratio of the running cost objective and the terminal cost objective should be scaled by $1/T$ and the dimension of $x \in \Re^n$ and $u \in \Re^m$:

$$\int_{t_0}^{t_f = t_0 + T} \frac{1}{nT} x^T Q x + \frac{1}{mT} u^T R u \;\; dt + x^T(t_f) S x(t_f).$$

  where Q, R, S are selected based on separate $x(t)$, $u(t)$ and $x(t_f)$ criteria. Additional relative scalings should be iteratively determined.
- If $R = \text{diag}[r_1, r_2, r_3]$ and after simulation, $|u_2|$ is too large, increase r_2;
- If after simulation, state x_3 is too large, modify Q such that $x^T Q x \leftarrow x^T Q x + \gamma x_3^2$ etc.
- If performance is related to frequency, use frequency weighting (see below).

6.7 Frequency Shaping

The original LQ problem is specified in the time domain. The cost function is basically the L_2 norms of the control, and of $z = Q^{\frac{1}{2}} x$. In many situations, it is more advantageous to specify the criteria in frequency domain. For example, it might be more costly to utilizing control effort that has high bandwidth; or if we know that disturbances to the system lie within a narrow bandwidth. Control that achieve robustness are also more easily specified in the frequency domain (e.g. in loop shaping concepts).

We begin with the **Parseval Theorem** which states that for a squared integrable function $h(t) \in \Re^p$ with $\int_{-\infty}^{\infty} h^T(t) h(t) dt < \infty$,

$$\int_{-\infty}^{\infty} h^T(t) h(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} H^*(jw) H(jw) dw \tag{6.24}$$

where $H(jw)$ is the fourier transform or as $H(s = jw)$, i.e. the Laplace transform of $h(t)$ evaluated at $s = jw$. $H^*(jw)$ denotes the conjugate transpose of $H(jw)$. Hence, for $H(s)$ with real coefficient, $H^*(jw) = H(-jw)^T$.

Parseval theorem states that the energy in the signal can be evaluated either in the frequency or in the time domain.

So, suppose that we want to optimize the criteria in the frequency domain as:

$$J(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(jw)Q_1^*(jw)Q_1(jw)X(jw) + U^*(jw)R_1^*(jw)R_1(jw)U(jw)\ dw \tag{6.25}$$

This says that the state and control weightings are given by

$$Q(w^2) = Q_1^*(jw)Q_1(jw); \qquad R(w^2) = R_1^*(jw)R_1(jw).$$

If we define $X_1(jw) = Q_1(jw)X(jw)$, $U_1(jw) = R_1(jw)U(jw)$, then

$$J(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1^*(jw)X_1(jw) + U_1^*(jw)U_1(jw)\ dw$$

Now, apply Parseval Theorem in reverse,

$$J(u) = \int_{-\infty}^{\infty} x_1^T(t)x_1(t) + u_1^T(t)u_1(t)\ dt. \tag{6.26}$$

If we know the dynamics of x_1 and u_1 is the control input, then we can solve using the standard LQ technique.

We express the filters $Q_1(s)$ and $R_1(s)$ as filters (e.g. low pass and high pass) with the actual state and input of the system $x(t)$ and $u(t)$ as inputs, and frequency weighted state $x_1(t)$ and $u_1(t)$ as outputs:

$$Q_1(s) = C_Q(sI - A_Q)^{-1}B_Q + D_Q \tag{6.27}$$
$$R_1(s) = C_R(sI - A_R)^{-1}B_R + D_R \tag{6.28}$$

which says that in the time domain:

$$\dot{z}_1 = A_Q\, z_1 + B_Q\, x \tag{6.29}$$
$$x_1 = C_Q\, z_1 + D_Q\, x \tag{6.30}$$

and similarly,

$$\dot{z}_2 = A_R\, z_2 + B_R\, u \tag{6.31}$$
$$u_1 = C_R\, z_2 + D_R\, u. \tag{6.32}$$

Hence we can define an augmented plant:

$$\frac{d}{dt}\begin{pmatrix} x \\ z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} A & 0 & 0 \\ B_Q & A_Q & 0 \\ 0 & 0 & A_R \end{pmatrix}\begin{pmatrix} x \\ z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} B \\ 0 \\ B_R \end{pmatrix} u(t)$$

or with $\bar{x} = [x; z_1; z_2]$, etc.

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u.$$

Since

$$u_1 = \begin{pmatrix} 0 & 0 & C_R \end{pmatrix}\bar{x} + D_R u$$
$$x_1 = \begin{pmatrix} D_Q & C_Q & 0 \end{pmatrix}\bar{x}$$

the cost function Eq.(6.26) becomes:

$$J(u) = \int \begin{pmatrix} \bar{x}^T & u^T \end{pmatrix} \begin{pmatrix} Q_e & N^T \\ N & R_e \end{pmatrix} \begin{pmatrix} \bar{x} \\ u \end{pmatrix} dt \tag{6.33}$$

where

$$Q_e = \begin{pmatrix} D_Q^T D_Q & D_Q^T C_Q & 0 \\ C_Q^T D_Q & C_Q^T C_Q & 0 \\ 0 & 0 & C_R^T C_R \end{pmatrix}$$

$$N = \begin{pmatrix} 0 \\ 0 \\ C_R^T D_R \end{pmatrix}; \qquad R_e = D_R^T D_R.$$

Eq.(6.33) is still not in standard form yet because of the off diagonal block N. We can convert Eq.(6.33) into the standard form if we consider:

$$u(t) = -R_e^{-1} N \bar{x} + v \tag{6.34}$$

The integrand in Eq.(6.33) becomes:

$$\begin{aligned} &\begin{pmatrix} \bar{x}^T & v^T \end{pmatrix} \begin{pmatrix} I & -N^T R_e^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} Q_e & N^T \\ N & R_e \end{pmatrix} \begin{pmatrix} I & 0 \\ -R_e^{-1} N & I \end{pmatrix} \begin{pmatrix} \bar{x} \\ v \end{pmatrix} \\ =&\begin{pmatrix} \bar{x}^T & v^T \end{pmatrix} \begin{pmatrix} Q_e - N^T R_e^{-1} N & 0 \\ 0 & R_e \end{pmatrix} \begin{pmatrix} \bar{x} \\ v \end{pmatrix} \end{aligned}$$

Then, define

$$\bar{Q} = Q_e - N^T R_e^T N, \qquad \bar{R} = R_e \tag{6.35}$$

and new state dynamics:

$$\dot{\bar{x}} = (\bar{A} - \bar{B} R_e^{-1} N)\bar{x} + \bar{B} v \tag{6.36}$$

and cost function,

$$J(v) = \int \bar{x}^T \bar{Q} \bar{x} + v^T \bar{R} v \; dt. \tag{6.37}$$

Eqs.(6.36)-(6.37) are then in the standard LQ format.

The stabilizability and detectability conditions are now needed for the the augmented system (what are they?).

6.8 Solution to the ARE via the Hamiltonian Matrix

For the infinite time horizon LQ problem, with (A, B) stabilizable and (A, C) detectable, P_∞ must satisfy the Algebraic Riccati Equation (ARE):

$$A^T P_\infty + P_\infty A - P_\infty B R^{-1} B^T P_\infty + Q = 0. \tag{6.38}$$

This is a nonlinear algebraic quadratic matrix equation. There are generally multiple solutions. Is it possible to solve this without integrating the CTRDE (6.13)?

Recall that the solution $P(t)$ can be obtained from the matrix Hamilton equation:

$$\begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix}}_{\text{Hamiltonian Matrix - } H} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \tag{6.39}$$

with boundary conditions: $X_1(t_f)$ invertible and $X_2(t_f) = SX_1(t_f)$ so that $P(t) = X_2(t)X_1^{-1}(t)$.

Denote the 2n eigenvalues and 2n eigenvectors of H by respectively:

$$\{\lambda_1, \lambda_2, \ldots, \lambda_{2n}\}, \qquad \{e_1, e_2, \ldots, e_{2n}\}$$

Let us choose n pairs of these:

$$\Lambda = \operatorname{diag}\{\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{in}\}, \qquad \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} e_{i1} & e_{i2} & \ldots & e_{in} \end{pmatrix},$$

Proposition 6.8.1 *Let $P_\infty := GF^{-1}$ where the columns of $\begin{pmatrix} F \\ G \end{pmatrix} \in \Re^{2n\times n}$ are n of the eigenvectors of H. then, P_∞ satisfies the Algebraic Riccati Equation* (6.38).

Proof: We know that $P(t) = X_2(t)X_1^{-1}(t)$ where $X_1(t)$ and $X_2(t)$ satisfy the Hamiltonian differential equation (6.39). For $P_\infty = GF^{-1}$ to satisfy Eq.(6.38), one needs only show that $\dot{P}(t) = 0$ when $X_1(t) = F$ and $X_2(t) = G$. This is so because

$$\begin{pmatrix} F \\ G \end{pmatrix} \Lambda = \underbrace{\begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix}}_{\text{Hamiltonian Matrix - } H} \begin{pmatrix} F \\ G \end{pmatrix}. \tag{6.40}$$

so that

$$\begin{aligned} \dot{P} &= G \left.\frac{dX_1^{-1}}{dt}\right|_F + \left.\frac{dX_2}{dt}\right|_G F^{-1} \\ &= -GF^{-1}F\Lambda F^{-1} + G\Lambda F^{-1} = 0. \end{aligned}$$

◇

This proposition shows that there are “$2n$ choose n” (i.e. 2n!/n!) solutions of P_∞, depending on which n of the $2n$ eigenvectors of H are picked to define F and G.

However, we know that the closed loop system matrix:

$$A_c = A - BR^{-1}B^T P_\infty = A - BR^{-1}B^T GF^{-1}.$$

must be stable if (A, B) is stabilizable and (A, C) is detectable.

Proposition 6.8.2 *Suppose that (A, B) is stabilizable and (A, C) is detectable. Then, the eigenvalues of H are symmetrically located across the imaginary and real axes with no eigenvalues on the imaginary axis.*

Proof: Consider a invertible coordinate transformation

$$T = \begin{pmatrix} I & 0 \\ P_\infty & I_n \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} I & 0 \\ -P_\infty & I_n \end{pmatrix}.$$

Hence,

$$T^{-1}HT = \begin{pmatrix} \underbrace{A - BR^{-1}B^T P_\infty}_{A_c} & -BR^{-1}B^T \\ 0 & \underbrace{-(A - BR^{-1}B^T P_\infty)^T}_{A_c^T} \end{pmatrix}$$

Since $T^{-1}HT$ and H share the same eigenvalues, this shows that H contains the eigenvalues of A_c as well as of $-A_c^T$. Hence, n eigenvalues of H must lie on the closed RHP, and n eigenvalues lie on the closed LHP. In other words, the eigenvalues of H are symmetrically located about both the real and imaginary axes.

Further, A_c (and hence H) cannot have any eigenvalues on the imaginary axis. For, otherwise, the optimal cost will be infinite. Hence, H must have n eigenvalues on the open LHP, and n on the open RHP. ◇

Proposition 6.8.3 *Suppose that (A, B) is stabilizable and (A, C) is detectable. The steady state solution of the CTRDE is the $P_\infty = GF^{-1}$ where $\begin{pmatrix} F \\ G \end{pmatrix}$ are chosen to consist of the n eigenvectors that correspond to the stable eigenvalues of H.*

Proof: Since

$$A_c = A - BR^{-1}B^T P_\infty = A - BR^{-1}B^T GF^{-1}$$

$$A_c F = [A, BR^{-1}B^T]\begin{pmatrix} F \\ G \end{pmatrix} = F\Lambda$$

where the last equality is obtained from Eq.(6.40). Hence, diag(Λ) consists of the eigenvalues of A_c, and columns of F are the eigenvectors. Since (A, B) and (A, C) are stabilizable and detectable, A_c is stable. Thus, Λ must have negative real parts. ◇

Remark Integrating the Hamiltonian matrix is not a good idea, either in forward time or in reverse time, since either way will be unstable. Integrating the Riccati backwards in time is more reliable. The Hamiltonian matrix is useful for solving for the solution to the ARE though, via its eigenvalues and eigenvectors.

6.9 Return Difference Equality and Eigenvalues of LQ system

Let $\Phi(s) = (sI - A)^{-1}$ and $G(s) = C(sI - A)^1 B = C\Phi(s)B$ where $Q = C^T C$.

Let the optimal feedback be: $u = -R^{-1}B^T Px(t)$ where P is the steady state solution of the Riccati equation, and $K = R^{-1}B^T P$ is the feedback gain.

Proposition 6.9.1 *The LQ optimal feedback system satisfies the following so called Return Difference Equality:*

$$(I + K\Phi(-s)B)^T R(I + K\Phi(s)B) = R + G^T(-s)G(s).$$

$I + K\Phi(s)B$ is known as the Return Difference as it computes difference of the signal before and after the feedback loop.

Proof: From the Alegbraic Riccati Equation Eq.(6.38) and by adding and subtract sP,

$$(-sI - A)^T P_\infty + P_\infty(sI - A) + P_\infty BR^{-1}B^T P_\infty = C^T C$$

Multiplying on the left by $B^(- sI - A^T)^{-1}$ and on the right by $(sI - A)^{-1}B$ and noting that $RK = B^T P$,

$$
\begin{aligned}
&B^T(-sI - A^T)^{-1}K^T R + RK(sI - A)^{-1}B + B^T(-sI - A^T)^{-1}K^T RK(sI - A)^{-1}B \\
=&B^T(-sI - A^T)^{-1}C^T C(sI - A)^{-1}B = G^T(-s)G(s)
\end{aligned}
$$

By grouping the terms on left hand side into a quadratic form:

$$[I + B^T(sI - A^T)^{-1}K^T]R[I + K(sI - A)^{-1}B] = G^T(-s)G(s)$$

◇

Notice that $L(s) = K\Phi(s)B$ (dimension of m = number of input) is like the loop gain of the closed loop system; and $G(s)$ is the open loop (uncontrolled) system. Classically, $I + L(s)$ is the difference between a signal entering the loop and itself after going round the loop once. It is called the Return Difference.

6.9.1 Robustness of LQ

The return difference equality gives a robustness property of the system. Plotting $L(j\omega)$ as in Nyquist plot, the closed loop system is stable if $L(j\omega)$ has the appropriate number of encirclement of the -1 point. Since the nominal LQ system is stable, the Nyquist plot tells us how much $L(j\omega)$ can be perturbed without changing encirclement of the $(-1, 0)$ point.

For the single input system, let $r = R$. The return difference equality says:

$$|1 + L(j\omega)|^2 = 1 + \frac{1}{r}|G(j\omega)|^2 \geq 1.$$

Hence, $L(j\omega)$ is separated from $(-1, 0)$ by a disk of radius 1 centered at $(-1, 0)$.

This implies the following robustness properties of LQ system:

- Infinite positive gain margin
- 50% negative gain margin
- 60 degree phase margin.

Simultaneous change in phase and magnitude can reduce these robustness results.

6.9.2 Discrete time systems - robustness etc.

There is a similar result for discrete time LQ which says that:

$$[I + K\Phi(z^{-1})B]^T(R + B^T PB)[I + K\Phi(z)B] = R + G^(z^{-1})G(z)$$

from which we have for (Single Input systems) the following robustness properties:

- Let $L(z) = K\Phi(z^{-1})B$,

$$\|1 + L(e^{jw})\| \geq \sqrt{\frac{R}{R + B^T PB}}$$

- The tolerable loop % loop gain change:

$$\frac{100}{1 + \sqrt{[R/(R + B^T PB)]}} < \%loopgainchange < \frac{100}{1 - \sqrt{[R/(R + B^T PB)]}}$$

- Phase margin $> 2sin^{-1}(0.5\sqrt{R/(R + B^T PB)})$.

6.9.3 Symmetric Root Locus

Lemma 6.9.2 *The determinant of the return difference satisfies the following relation relating closed loop poles to the open loop poles.*

$$det[I + K\Phi(s)B] = \frac{det(sI - A + BK)}{det(sI - A)}$$

Let $\beta(s) = det(sI - A + BK)$ be the closed loop characteristic polynomial; and $\alpha(s) = det(sI - A)$ be the open loop characteristics polynomial.

Using the above lemma, by taking the determinant of the return difference equality gives:

$$\frac{\beta(-s)\beta(s)}{\alpha(-s)\alpha(s)} = det(I + R^{-1}G^T(-s)G(s))$$

For an r output system, $G(s) = C(sI - A)^{-1}B = \frac{CAdj(sI-A)B}{\alpha(s)} = \frac{1}{\alpha(s)}\Phi(s)$ where $\Phi(s) = \begin{pmatrix} \Phi_1(s) \\ \Phi_2(s) \\ \vdots \\ \Phi_r(s) \end{pmatrix}$.

Rewriting $\Phi^T(-s)\Phi(s) = m(-s)m(s)$ (a scalar polynomial), we have

$$\beta(-s)\beta(s) = \alpha(-s)\alpha(s) + \frac{1}{r}m(s)m(-s)$$

Notice that $m(s)m(-s)$ is the numerator $G^T(s)G(s)$. In the single output case, $m(s)$ IS the numerator of $G(s)$ and its roots are the zeros of $G(s)$.

This relationship reminds us of the root locus technique of finding the closed loop poles based on open loop poles and zeros. The one exception is that we need to include both the open loop poles and zeros and their reflections about the imaginary axis. Also, the root locus is for the closed loop poles and the reflection across the imaginary axis.

We have the following results:

- When r is large, i.e. $1/r$ is small (control is expensive), the closed loop poles approach the stable open loop poles or the negative of the unstable open loop poles.
- When r is small, i.e. $1/r$ is large, (control is cheap), as many closed loop poles as number of open loop zeros are close to stable open loop zeros or the negative of the non-minimum phase open loop zeros.
- In the cheap control case, the remaining poles approach infinity in a manner such that they and their reflections across the imaginary axis have asymptotes that are evenly distributed.

This motivates the design rule:

1. Choose C in $Q = C^TC$ such that $n-1$ zeros of $G(s) = C(sI - A)^{-1}B$ are at the desired pole location.
2. Use cheap control $r \to 0$ to design LQ system so that $n-1$ poles approach these desired locations.
3. It can be shown that when $r \to 0$, $K \approx C/\sqrt{r}$. So that the loop gain is approximately $L(s) = \frac{1}{\sqrt{r}}C(sI - A)^{-1}B$. At high frequency $|L(j\omega)| \approx \frac{CB}{\sqrt{r}\omega}$
4. We can choose r to pick the bandwidth ω_c which is where $|L(j\omega_c)| \approx 1$. Thus, choose $r \approx CB/\omega_c$ where ω_c is the desired bandwidth.